COMMON FIXED POINT THEOREMS FOR WEAK COMPATIBLE MAPPINGS OF TYPE (A)

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ABSTRACT
In this paper, we prove common fixed point theorems for weak compatible mapping of type (A) for four mappings in a metric space which generalizes the results of Kang and Kim, Fisher, Jungck, Lohani and Badshah.
Keywords : weak compatible mappings of type (A), metric spaces.

1. Introduction and preliminaries:

In all generalizations of Jungck's theorems, families of commuting mappings have been considered. Rhoades-Sessa-Khan [8] improved the previous results by assuming weak commutativity. In 1992 Jungck-Murthy-Cho [3] introduced the concept of compatible mappings of type [A] in metric spaces and improved the results of various authors. We exploit the idea of weak compatible mappings of type (A) in metric space as used by Pathak -Kang – Baek[[6],[7]] in menger and 2-metric spaces respectively which is equivalent to the concept of compatible and compatible mappings of type(A) under some conditions.

This concept is more general than that of weak commutativity, compatible and compatible of type (A). In this paper we prove a common fixed point theorem for weak compatible mapping of type (A) for four mappings in a metric space which generalizes the result of Fisher [1], Lohani and Badshah [5] and Kang and Kim.

1.1. Definition [3]. Let f, g: (X, d) $\to$ (X, d) be mappings. Then they are said to be **compatible of type (A)** if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) = 0 \text{ and } \lim_{n\to\infty} d(gfx_n, ffx_n) = 0,$$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

1.2. Definition [6]. Two mappings f, g: X $\to$ X, where X is a metric space are said to be **weak compatible of type (A)** if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \le \lim_{n\to\infty} d(gfx_n, ggx_n) \text{ and}$$

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \le \lim_{n\to\infty} d(fgx_n, ffx_n)$$

Whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Now we give some propositions before presenting our result which are very useful.

1.1 Proposition. Every pair of compatible mappings of type (A) is weak compatible of type (A).

Proof. Suppose that f and g are compatible mappings of type (A). Therefore,

$$0 = \lim_{n\to\infty} d(fgx_n, ggx_n) \le \lim_{n\to\infty} d(gfx_n, ffx_n) \text{ and}$$

$$0 = \lim_{n\to\infty} d(gfx_n, ffx_n) \le \lim_{n\to\infty} d(fgx_n, ffx_n)$$

Which shows that the pair {f,g} is weak compatible of type (A).

1.2. Proposition. Let f and g be continuous mappings of a metric space (X,d) into self. If f and g are weak compatible of type (A), then they are compatible of type (A).

Proof: Suppose that f and g are weak compatible of type (A). Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$

Since f and g are continuous mappings, then we have

$\lim_{n\to\infty} d(fgx_n, ggx_n) \le \lim_{n\to\infty} d(gfx_n, ggx_n) = d(gt, gt) = 0$ and

$\lim_{n\to\infty} d(gfx_n, ffx_n) \le \lim_{n\to\infty} d(fgx_n, ffx_n) = d(ft, ft) = 0$

Therefore, f and g are compatible mappings of type (A). This completes the proof.

1.3. Proposition. Let f and g be weak compatible mappings of type (A) from a metric space (X,d) into itself. If one of f and g is continuous, then they are compatible.

Proof. Without loss of generality, let us suppose that g is continuous

Let $\{X_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \text{ for time } t \in X$$

Since g is continuous, we have

$$\lim_{n\to\infty} gfx_n = gt = \lim_{n\to\infty} ggx_n$$

Now $d(gfx_n, fgx_n) \le d(gfx_n, ggx_n) + d(ggx_n, fgx_n)$

$$\le 0 + d(ggx_n, fgx_n) = d(ggx_n, fgx_n)$$

Since (f,g) are weak compatible of type (A). Therefore, we have

$$\lim_{n\to\infty} d(gfx_n, fgx_n) \le \lim_{n\to\infty} d(fgx_n, ggx_n)$$

$$\le \lim_{n\to\infty} d(gfx_n, ggx_n) \le 0$$

Therefore f and g are compatible.

1.4. Proposition. Let f and g be continuous mappings of (X,d) into itself. If f and g are compatible, then they are compatible of type (A). As a direct consequence of Propositions 1.2 and 1.3, we have the following proposition:

1.5. Proposition. Let f and g be continuous mapping from a metric space (X, d) into itself. If f and g are compatible then they are weak compatible of type (A).

Next we give some properties of weak compatible mappings of type (A).

1.6. Proposition. Let f and g be weak compatible maps of type (A) from metric space (X,d) into itself and let $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$ then we have the following :

(1) $\lim_{n\to\infty} fgx_n = gt$ If g is continuous at t.

(2) $\lim_{n\to\infty} gfx_n = ft$ if f is continuous at t.

(3) $gft = fgt$ and $ft = gt$ if g and f are continuous at t.

1.7 Proposition. Let f and g be continuous mappings from a metric space (X,d) into itself. Then

(i) f and g are compatible of type (A) if and only if they are weak compatible of type (A).

(ii) f and g are compatible if and only if they are weak compatible of type (A).

2. MAIN RESULT

In this section we prove common fixed point theorems for weak compatible mappings of type (A) for four mappings in metric spaces.

Let f, g, S and T be self mappings from the metric space (X,d) into itself satisfying the following conditions:

(2.1) $f(X) \subseteq T(X), g(X) \subseteq S(X)$

(2.2) $d(fx, gy) \le h \max.\left\{d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)], d(Sx, Ty)\right\}$ for all x,y in X, where $0 \le h \le 1$.

Then for any arbitrary point x_0 in X by (2.1), we choose a point $x_1 \in X$ such that $Tx_1 = fx_0$ and for this point x_1 we can choose a point $x_2 \in X$ such that $Sx_2 = gx_1$ and so on inductively, we can define a sequence $\{y_n\}$ in X such that

(2.3) $y_{2n+1} = Tx_{2n+1} = fx_{2n}$ and $y_{2n} = Sx_{2n} = gx_{2n-1}$

In 1988, Jungck [2] proved a lemma which is very useful in proving common fixed point theorem in metric spaces.

2.1. Lemma [2]. Let f, g, S and T be mappings from a metric space {X, d} into itself satisfying the conditions (2.1) and (2.2). Then the sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in X.

2.2. Lemma. Let f, g, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (2.1), (2.2), (2.3) and the following:

(2.4) $f(X) \cap g(X)$ is a complete subspace of X.

Then pairs (f, S) and (g, T) have a coincidence point in X

Proof: By Lemma 2.1 the sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in $f(X) \cap g(X)$. Since $f(X) \cap g(X)$ is a complete subspace of X. So $\{y_n\}$ converges to a point w (say), in $f(X) \cap g(X)$. On the other hand, since the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ are also Cauchy sequences in $f(X) \cap g(X)$. So they also converge to the same limit w. Hence there exist two points u, v in X such that Au=w and Bv=w, respectively. By (2.2) we obtain

$$d(Su, y_{2n+1}) = d(Su, fx_{2n}) \le h \max\left\{ d(Su, Tu), d(fx_{2n}, gx_{2n+1}) \frac{1}{2}[d(Su, gx_{2n+1}) + d(Tx_{2n+1}, fu)], d(fu, Tx_{2n+1}) \right\}$$

Letting limit as $n \to \infty$ we have $d(Su, w) \le h \max\left\{ d(Su, w), d(w, w), \frac{1}{2}[d(Su, w) + d(w, fu)], d(fu, w) \right\}$

$= h\, d(Su, w)$, a contradiction.

Hence fu=w=Su Similarly, we can show that v is also a coincidence point of g and T.

2.3. Lemma. Let f and S be weak compatible mappings of type (A) from a metric space (X,d) into itself. If fu=Su for some $u \in X$, then $Sfu = SSu = ffu = fSu$

Proof. Let $\{x_n\}$ be a sequence In X defined by $x_n = u$, $n = 1,2,3$ and $Su = fu$.

Now we have $\lim_{n\to\infty} Sx_n = Su = \lim_{n\to\infty} fx_n$

Since S and f are weak compatible mappings of type (A), we have

$$d(Sfu, ffu) = \lim_{n\to\infty} d(Sfx_n, ffx_n)$$
$$\le \lim_{n\to\infty} d(fSx_n, ffx_n) = 0$$

Hence Sfu=ffu. Therefore, Sfu=SSu=ffu=fSu.

In 1992, S. M. Kang and Y.P. Kim [4] and in 1998 P.C. Lohani and V.H. Badshah[5] proved a common fixed theorem for continuous compatible mappings in metric spaces as follows:

2.4. Theorem [4]. Let f, g, S and T be self mappings from a metric space (X,d) into itself satisfying (2.1), (2.2), (2.3) and following:

(2.5) One of f, g, S and T is continuous.

(2.6) Pairs (f, S) and (g, T) are compatible on X.

Then f, g, S and T have a unique common fixed point in X.

We improve the above result by proving a common fixed point theorem for weaker hypothesis as follows:

2.5. Theorem. Let f, g, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (2.1), (2.2), (2.3) and the following:

(2.7) the pairs (f, S) and (g, T) are weak compatible mappings of type (A).

Then f, g, S and T have a unique common fixed point in X .

Proof. By Lemma 2.2, there exist two points u, v in X such that Su=fu=w and Tv=gv=w, respectively. Since f and S are weak compatible of type (A), then by Lemma 2.3. Sfu=SSu=ffu=fSu, which implies that Sw=fw. Similarly we have Tw=gw. Now, we prove that Sw=w. Suppose $Sw \ne w$, then by (2.2) we have

$$d(Sw, y_{2n+1}) = d(Sw, Tx_{2n+1}) \le h \max \left\{ \begin{array}{l} d(Sw, fw), d(Tx_{2n+1}, gx_{2n+1}), \\ \frac{1}{2}(d(Sw, gx_{2n+1}) + d(Tx_{2n+1}, fw))d(fw, Tx_{2n+1}) \end{array} \right\}$$

Proceeding to limit as $n \to \infty$, we have

$$d(Sw,w) \le h \max\left\{d(Sw,w), d(w,w), \frac{1}{2}[d(Sw,w)+d(w,fw)]d(fw,w)\right\}$$

$= hd(Sw,w)$ a contradiction, Hence, Sw=w=fw

Similarly, we have Tw=w=gw. This means that w is a common fixed point of f, g, S and T.

Uniqueness:

Let $z \ne u$ be another common fixed point of f, g, S and T.

Then we have

$$d(u,z) = d(Su,Tz) \le h \max\left\{d(Su, fu), d(Tz, gz)\frac{1}{2}[d(Su, gz) + d(Tz, fu)], d(fu,Tz)\right\} = hd(Su,Tz)$$

$= hd(u,z)$ a contradiction. Hence $u = z$

Now we give an example in support of our theorem.

Example. Let x= [0, 1] with the Euclidean metric d.

Define f, g, S and T: $X \to X$ by

$fx = x^3, gx = x^2, Sx = 2x^6 - 1$ and $Tx = 2x^4 - 1$ for all x in X.

Now f(X)=g(X)=S(X)=T(X)=X. Moreover, since

$d(fx_n, Sx_n) = |2x_n^3 + 1||x^3 - 1| \to 0$, Iff $x_n \to 1$

$\lim_{n\to\infty} d(fSx_n, Sfx_n) = \lim_{n\to\infty} 6x_n^6(x_n^6 - 1)^2 = 0$ as $x_n \to 1$

Also, since $d(gx_n, Tx_n) = (2x_n^2 + 1)|x_n^2 - 1| \to 0$ iff $x_n \to 1$

$\lim_{n\to\infty}(gTx_n, Tgx_n) = \lim_{n\to\infty} 2(x_n^4 - 1)^2 = 0$ as $x_n \to 1$

Clearly (f, S) and (g, T) are weak compatible mappings of type (A).

Further, we obtain

$$d(fx, gy) \le \frac{1}{4} d(Sx,Ty) \le \frac{1}{4} \max\left\{d(fx,Sx), d(gy,Ty), d(Sx,Ty)\frac{1}{2}(d(fx,Ty) + d(gy,Sx))\right\}$$

Since $d(Sx,Ty) = 2|x^3 - y^2||x^3 + y^2| \ge 4d(fx, gy)$ for all $x, y \in X$

Therefore all the conditions of the theorem (2.2) are satisfied and one is the unique common fixed point.

2.6. Remark.

Our theorem improves the result of Kang and Kim [4] in two aspects. Firstly our theorem does not require the mappings to be continuous; secondly we prove the result for weak compatible mappings of type (A) instead of compatible mappings.

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